Phase synchronization in tilted inertial ratchets as chaotic rotators

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The phenomenon of phase synchronization for a particle in a periodic ratchet potential is studied. We consider the deterministic dynamics in the underdamped case where the inertia plays an important role since the dynamics can become chaotic. The ratchet potential is tilted due to a constant external force and is rocking by an external periodic forcing. This potential has to be tilted in order to obtain a rotator or self-sustained nonlinear oscillator in the absence of the external periodic forcing; this oscillator then acquires an intrinsic frequency that can be locked with the frequency of the external driving. We introduced an instantaneous linear phase, using a set of discrete time markers, and the associated average frequency, and show that this frequency can be synchronized with the frequency of the driving. We calculate Arnold tongues in a two-dimensional parameter space and discuss their implications for the chaotic transport in ratchets. We show that the local maxima in the current correspond to the borders of these Arnold tongues; in this way we established a link between optimal transport in ratchets and phase synchronization. © *2008 American Institute of Physics*. [DOI: 10.1063/1.3043423]

Synchronization in chaotic dynamical systems has been of major interest both from a fundamental point of view and due to its potential applications in a wide variety of systems. In particular, phase synchronization has been ubiquitous to study chaotic oscillators and rotators. A paradigmatic example can be the underdamped pendulum with a constant torque and a periodic driving. Here, we consider a similar system: an inertial tilted deterministic ratchet that acts as a rotator or self-sustained oscillator. The ratchet potential is an asymmetric spatially periodic potential that can induce directed motion by rectifying time-periodic forces of zero average. The main quantity of interest is the average velocity (current) of a particle in this rocking ratchet potential. In this paper, we introduce a linear phase for this rotator, through a set of discrete time events, and calculate the average frequency and velocity. We observed a series of steps in the current that indicates frequency locking, and we obtained the corresponding Arnold tongues in the parameter space given by the amplitude and frequency of the forcing. A connection between optimal transport in ratchets and these regions of synchronization is established.

I. INTRODUCTION

In recent times the phenomenon of synchronization in nonlinear dynamics has received considerable attention and has been used to understand a wide variety of topics in the physical sciences and biology.[1–5] In particular, phase synchronization has been established as a common formalism to treat nonlinear periodic, chaotic, and noisy oscillators.[1,4,6]

On the other hand, transport phenomena in asymmetric periodic potentials have been studied with the goal of understanding the role of nonlinearity in the rectification of unbiased nonequilibrium fluctuations. These so-called ratchets are usually modeled using an asymmetric nonlinear spatially periodic potential (ratchet potential), together with an external unbiased time-dependent force.[7–13]

Among the many kinds of ratchets studied recently, an important class refers to classical deterministic inertial ratchets in which the dynamics does not have any randomness or stochastic elements. Being a nonlinear system, in some cases its deterministic dynamics can exhibit chaotic motion. This is the case when we consider inertial effects, for instance, in a one-dimensional rocking ratchet.[14–17] These ratchets can be modeled by a particle with inertia and friction on a one-dimensional asymmetric ratchet potential and acted upon by a harmonic time-dependent force of zero average. In this case, a surprising phenomenon appears: current reversals.[14–25] These reversals of the average velocity where explained in Ref. 15 by establishing a connection between the current and the bifurcation diagram as a function of a control parameter. The study of inertial deterministic ratchets has acquired importance due to recent experiments on vortex and SQUID ratchets that reveal, among other things, the effect of current reversals.[26–38]

In this paper we will study a deterministic ratchet in the underdamped regime in the presence of an external constant force. Thus, we have a tilted inertial ratchet that, for a tilt above a critical value, can act as a rotator or self-sustained oscillator with a characteristic frequency, even when an external periodic force is absent. The dynamics can be modeled as an inertial particle in a washboard potential that has been analyzed in many different situations, like phase dynamics in synchronization,[1,6] pendulum dynamics,[39] rotators,[1] superionic conductors,[40] optical potentials,[41] excitable systems,[42] diffusion,[43–50] charge density waves,[51] and Josephson junctions dynamics.[1,40,52] If this washboard potential is acted upon by an external driving force, it can exhibit additional phenomena like phase locking, hysteresis,[53] chaos,[54] and anomalous transport.[55–61]

We will analyze here the dynamics of an underdamped particle moving on a tilted ratchet potential that is rocked by a periodic external force and, in particular, the phenomenon of synchronization. We will consider a value of the constant force such that, in the absence of the periodic forcing, the tilt induced in the ratchet potential is above the critical tilt. In this situation the particle slides down the fixed washboard potential and moves through each period of the ratchet in a fixed time that defines the period τ_0 of this rotator or self-sustained oscillator; $\omega_0=2\pi/\tau_0$ is the characteristic frequency of the rotator. We will drive this self-sustained oscillator with an external harmonic force of period ω_D, and through a comparison between these two frequencies, we can define the synchronization properties of the system.

Recent studies consider the problem of synchronization of deterministic ratchets, but they deal with complete synchronization between two coupled ratchets,[62–66] or a network of ratchets.[67] On the other hand, the problem of anticipated synchronization between two unidirectional coupled ratchets with time delay has been studied in Ref. 68. For an overview of phase synchronization in time delay systems, see Refs. 69 and 70, and references therein.

In this work, instead, we are dealing with phase synchronization using a linear phase defined through a set of discrete time events. The case of phase synchronization for an overdamped tilted ratchet was studied previously in Ref. 71, whereas in this paper we are considering the inertial case. We will calculate Arnold tongues for this tilted inertial ratchet and discuss the blueprint of synchronization in its transport properties.

II. TILTED RATCHETS AS CHAOTIC ROTATORS

Let us consider the one-dimensional problem of a point particle driven by a harmonic external force in an asymmetric periodic ratchet potential. Two additional forces act on the particle: a dissipative force proportional to the velocity, and an external constant force. The equation of motion for this rocked deterministic tilted ratchet in the underdamped case is

$$m\ddot{x} + m\gamma\dot{x} + \frac{dV(x)}{dx} = F + F_D\cos(\omega_D t), \tag{1}$$

where m is the mass of the particle, γ is the friction coefficient, $V(x)$ is the asymmetric periodic ratchet potential, F is a constant force, F_D, and ω_D represent the amplitude and the frequency of the external driving force, respectively. The ratchet potential is given by

$$V(x) = V_0\left[C - \sin\frac{2\pi(x-x_0)}{L} - \frac{1}{4}\sin\frac{4\pi(x-x_0)}{L}\right], \tag{2}$$

where L is the periodicity of the potential, V_0 is the amplitude, and C is an arbitrary constant. The potential is shifted by an amount x_0 so that the minimum of the potential is located at the origin.[15]

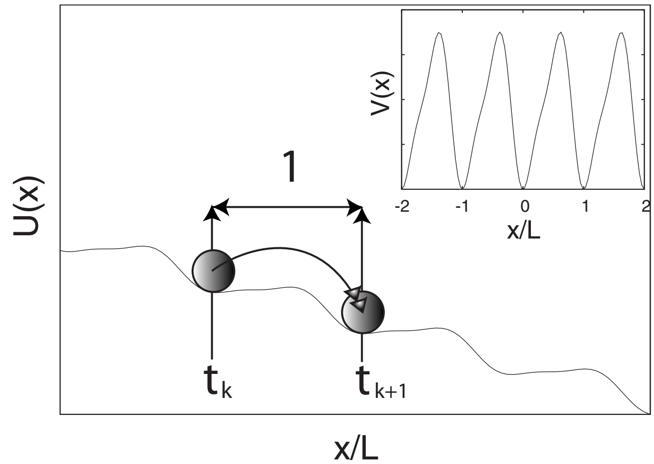

FIG. 1. An inertial particle sliding down a tilted ratchet potential indicating the dynamics that defines the discrete time events t_k. The difference between two consecutive events defines the natural period of this self-sustained oscillator or rotator. The ratchet potential without tilt is depicted in the inset.

We define the following dimensionless units: $x'=x/L$, $x_0'=x_0/L$, $t'=\gamma t$, $\omega_D'=\omega_D/\gamma$, $F'=F/mL\gamma^2$, $F_D'=F_D/mL\gamma^2$, $V'=V/mL^2\gamma^2$, and $V_0'=V_0/mL^2\gamma^2$.

After renaming the variables again without the primes, the dimensionless equation of motion becomes

$$\ddot{x} + \dot{x} + \frac{dV(x)}{dx} = F + F_D\cos(\omega_D t), \tag{3}$$

where the dimensionless potential can be written as

$$V(x) = V_0\left[C - \sin 2\pi(x-x_0) - \tfrac{1}{4}\sin 4\pi(x-x_0)\right] \tag{4}$$

and is shown in the inset of Fig. 1. Therefore now the periodicity of the potential is 1, $V(x+1)=V(x)$. The constant C is such that $V(0)=0$, and is given by $C=-(\sin 2\pi x_0 + 0.25\sin 4\pi x_0)$, where $x_0 \simeq -0.19$.[15]

We can rewrite the equation of motion Eq. (3) as

$$\ddot{x} + \dot{x} + \frac{\partial U(x,t)}{\partial x} = 0, \tag{5}$$

where $U(x,t)=V(x)-[F+F_D\cos(\omega_D t)]x$. When $F_D=0$, we have a tilted ratchet that obeys the equation of motion, $\ddot{x}+\dot{x}+dV(x)/dx=F$. The tilted (time-independent) washboard potential is, in this case, $U(x)=V(x)-Fx$, see Fig. 1. Thus, the ratchet becomes a rotator or self-sustained oscillator that has a characteristic frequency ω_0, that can be obtained directly by integrating this equation of motion.

In what follows we will introduce the concept of a phase variable that we will use to characterize the phenomenon of synchronization. To define a phase variable we need first to obtain a discrete process from the continuous dynamics by introducing discrete time events. These discrete times t_k can be defined as the times when the particle is at the discrete positions $x_k=k$, which correspond to the minima of the ratchet potential without tilt. Here $k=0,1,2,\ldots$. In Fig. 1 we show the washboard potential and these discrete times. Having obtained these set of events, we can define an instantaneous linear phase for the rotator as[1,6]

$$\phi(t) = 2\pi \frac{t - t_k}{t_{k+1} - t_k} + 2\pi k, \quad t_k \leq t < t_{k+1}. \tag{6}$$

This linear phase is valid in the indicated interval and is a piecewise linear function of time that increases by 2π each time the particle crosses the dimensionless position $x_k = k$.

Given this phase, we can define the instantaneous frequency of the rotator as $\omega(t) = \dot{\phi}(t)$, and the average frequency as

$$\langle \omega \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \omega(t) dt = \lim_{T\to\infty} \frac{1}{T} [\phi(T) - \phi(0)]. \tag{7}$$

We choose $t_0 = 0$, and thus $\phi(0) = 0$, without loss of generality. The limit above can be written as

$$\langle \omega \rangle = \lim_{k\to\infty} \frac{\phi(t_k)}{t_k} = 2\pi \lim_{k\to\infty} \frac{k}{t_k}. \tag{8}$$

This is the simplest way to calculate the average frequency; we count the number of jumps (given by k) and divide by the total time span t_k.

The other relevant quantity is the average velocity or current in the ratchet. In order to evaluate this current we have to calculate the number k of unit periods that the particle crosses to the right, denoted by N_k^R, and the number of crossings to the left, given by N_k^L. The total number of periods traversed on the ratchet is given by $N_k^T = N_k^R + N_k^L$. The difference $N_k = N_k^R - N_k^L$ indicates that during the time t_k the particle has covered the distance $x_k = N_k$.

Therefore, the average velocity (current) is given by

$$\langle v \rangle = \lim_{k\to\infty} \frac{N_k}{t_k}. \tag{9}$$

In the case when $N_k^L = 0$, that is, when there are no jumps to the left, we have $N_k = N_k^R = k$. Thus,

$$\langle v \rangle = \lim_{k\to\infty} \frac{k}{t_k} = \frac{1}{2\pi} \langle \omega \rangle. \tag{10}$$

In the particular case of a tilted ratchet without external forcing, $F_D = 0$, the average frequency defined above coincides with the natural frequency of the rotator, that is, $\langle \omega \rangle = \omega_0$.

The above treatment is quite general and can be used even though this inertial ratchet can display a chaotic dynamics. The concept of a linear phase is very useful since it can be applied to the cases of periodic and chaotic oscillators,[1] chaotic rotators,[72] and oscillators in the presence of noise.[6]

III. NUMERICAL RESULTS

We solved numerically the equation of motion Eq. (3) using a fourth-order Runge–Kutta algorithm. Once we obtain the trajectory $x(t)$, we identify the set of discrete times t_k when the particle crosses the positions x_k. Using these markers, we calculate the average frequency using Eq. (8) and, after calculating the quantity N_k, we obtain the current, using Eq. (9). We will fix throughout the paper the amplitude of the ratchet potential as $V_0 = 1/2\pi$. With this value, the critical tilt to the right is $F_c^R \simeq 0.71$ and to the left $F_c^L \simeq -1.26$.

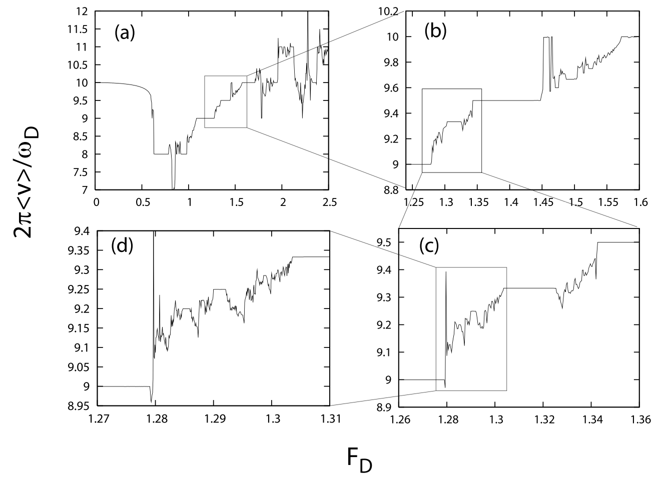

FIG. 2. The scaled average velocity $2\pi\langle v \rangle / \omega_D$ as a function of the amplitude of the periodic driving F_D. (a), (b), (c), and (d) correspond to successive magnifications of the current, showing a self-similar structure. In all cases we used $F = 1.0$ and $\omega_D / \omega_0 = 0.1$.

As discussed in Ref. 56, if we plot the average velocity as a function of the tilt F, in the case without periodic driving that corresponds to the fixed washboard potential $U(x) = V(x) - Fx$, we notice that the current is zero until we arrive at the critical tilt F_c^R to the right or to the critical tilt F_c^L to the left. This step of zero current is not centered on the origin, due to the asymmetry of the ratchet potential. For values greater than F_c^R we would have a finite current that increases monotonically with F. Of course, for values less than F_c^L we obtain a negative average velocity that decreases for negative values of the tilt. When the periodic driving is present, the current acquires a series of steps, as in the overdamped case,[71] for values of the current given by the ratio p/q, where p and q are integer numbers. Recall that in the case where all the jumps are to the right we show that $2\pi\langle v \rangle = \langle \omega \rangle$. Therefore, a rational value of $2\pi\langle v \rangle / \omega_D = p/q$ means that $\langle \omega \rangle = (p/q)\omega_D$ for a whole range of values of the tilt; a clear indication of frequency locking. Besides these steps, we can also have a more complex structure due to the chaotic dynamics that arises in this inertial case. We can even discern anomalous mobility for small values of the tilt, around the origin. This phenomenon has been analyzed in detail in Ref. 56.

In Fig. 2(a), we show the scaled average velocity $2\pi\langle v \rangle / \omega_D$ as a function of the amplitude of the periodic driving F_D, for a fixed tilt $F = 1$ and $\omega_D / \omega_0 = 0.1$. In Figs. 2(b)–2(d) we depict different magnifications of some portions of the current. We observe that the chaotic character of the dynamics introduce complex variations in the current that result in a self-similar structure. The steps that can be seen correspond to synchronization regions in parameter space, that is, Arnold tongues. In this figure we used the frequency ratio $\omega_D / \omega_0 = 0.1$; thus, its inverse gives exactly the value of the current (10 in this case). This can be seen in Fig 2(a) for small values of F_D. However, as we increase this amplitude, the current drops to a value of 8. This is a manifestation of the fact that the tongue 1:8 overlaps with the tongue 1:10. For even larger values of F_D, the current drops to a value of

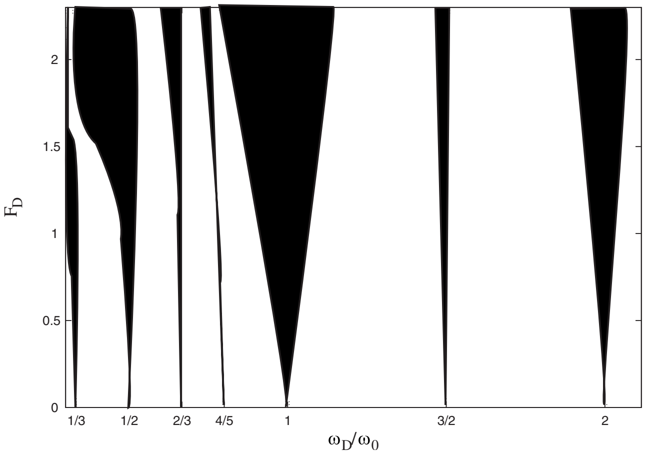

FIG. 3. If we plot the regions of synchronization in the parameter space, given by F_D and ω_D/ω_0, we obtained the so-called Arnold tongues, as depicted here. The tips of each of these tongues are located at the rational values p/q of the ratio between the driving frequency and the natural frequency of the rotor. Here p and q are integer numbers, the tilt is $F=1$, and the corresponding frequency is $\omega_0 \simeq 6.2$. Each tongue is labeled by a rational number p/q; whose inverse gives the value of the scaled average velocity $2\pi\langle v\rangle/\omega_D$ in that region of the parameter space.

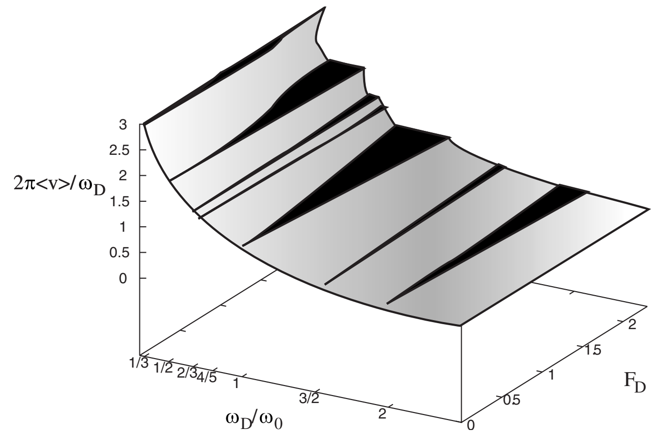

FIG. 4. The current or average velocity, properly scaled, acquires a structure of steps due to the phenomenon of phase synchronization of a particle in a rocking inertial ratchet. Here we show a three-dimensional plot of the scaled average velocity $2\pi\langle v\rangle/\omega_D$ as a function of F_D and ω_D/ω_0. The projection of this 3D plot of the current gives the Arnold tongues in the parameter space as in Fig. 3. In this case the value of the tilt is $F=1$ which correspond to $\omega_0 \simeq 6.2$.

7, due to an interaction with the tongue 1:7, and so on. Therefore, the complexity that arises due to the overlapping Arnold tongues translates directly into the flow of particles in the ratchet potential.

The regions of synchronization in parameter space F_D against ω_D/ω_0 are called Arnold tongues, and are depicted in Fig. 3. The tips of each of these tongues are located at rational values of ω_D/ω_0 and these rational numbers label each of the Arnold tongue. For small values of the amplitude of the driving force F_D the shape of the Arnold tongues is basically the same. The tip of each of these tongues is located at the rational values p/q, where p and q are integer numbers, and the borders grow linearly. However, for larger values of F_D each tongue gets distorted depending on the ratio p/q. This distortion is related with the fact that each tongue becomes wider as F_D increases; if, for example, neighboring tongues become wider, then the smaller tongues in between are squeezed out. Typically, the wider tongues are the ones with small denominators. What is important for the transport properties of a particle moving on the ratchet potential is that the value of its average velocity $2\pi\langle v\rangle/\omega_D$ for that region of the parameter space is precisely the inverse of the rational value that label each tongue.

To illustrate this connection between synchronization regions and transport in ratchets, in Fig. 4 we show a three-dimensional plot of the current as a function of the amplitude F_D and the frequency ratio ω_D/ω_0. The scaled average velocity of a particle in this rocking ratchet acquires a structure characterized by steps at rational values that are the inverse of the rationals that label each tongue. Therefore, the current, which is the main quantity in the physics of ratchet acquires a signature given by the phenomenon of phase synchronization.

If we make a slice of Fig. 4 for $F_D=1$ we obtain the average velocity as a function of the ratio ω_D/ω_0, as shown in Fig. 5. Here we can see clearly the steps in the current for integer values, that correspond to the widths of the Arnold tongues. Notice that the current scaled in this way is very large for small values of the driving frequency. However, this can be misleading since this effect is due the scaling, that is, the average velocity has a finite value but being divided by ω_D grows as we decrease this driving frequency.

In order to characterize more clearly the transport properties of a particle in this rocking inertial ratchet, we plot in Fig. 6, the average velocity but scaled now as $2\pi\langle v\rangle/\omega_0$. Scaled in this way, we avoid the effect discussed above in Fig. 5, that is, even for small values of the driving frequency ω_D we have a finite current. Therefore this figure represent more accurately the physics involved in the dynamics. Instead of plateaus, we now have straight lines, since we are scaling the current with a fixed frequency ω_0, instead of the running parameter ω_D. The slopes of these straight lines are given by the rational values associated with the Arnold

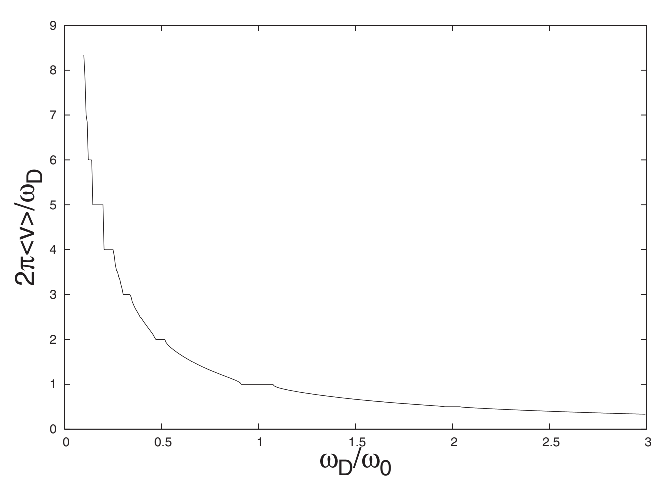

FIG. 5. Average velocity $2\pi\langle v\rangle/\omega_D$, as a function of the ratio ω_D/ω_0. The steps in the current correspond to the steps observed due to phase synchronization in Fig. 3. This figure corresponds to a slice of Fig. 4 for $F_D=1$. Here $F=1$ and $\omega_0 \simeq 6.2$.

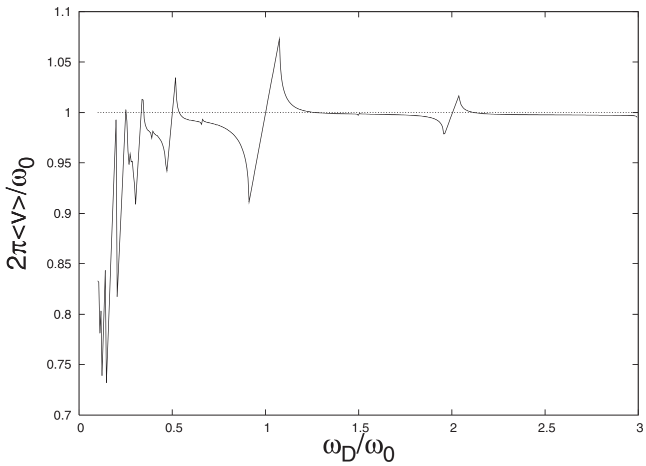

FIG. 6. Current or average velocity, scaled now as $2\pi\langle v\rangle/\omega_0$, as a function of the ratio ω_D/ω_0. Due to this scaling the plateaus shown in Fig. 5 become straight line segments that correspond to the steps observed due to phase synchronization. Notice that the peaks in the current are located at the right borders of the Arnold tongues in Fig. 3. In particular, the largest peak is associated with the tongue 1:1. We used here $F=1$, $F_D=1$, and $\omega_0\simeq 6.2$.

tongues. The higher the rational number labeling the tongue, the smaller the slope in the current. It is important to stress that the peak values of the average velocity correspond to the borders of the Arnold tongues in Fig. 3. Thus, when we are crossing an Arnold tongue the current increases linearly with ω_D/ω_0; at the right border of the tongue the current is maximal and outside the tongue starts to decrease until we arrive at the next tongue to increase linearly again, and so forth. At the right border of the Arnold tongue, labeled by $p/q=1$, the current has a maximum. The heights of the peaks in the current arise due to the combined effect of both the width of the tongues and the slope of the linear segments. In this way we show that there is a connection of the peak values in the current and the synchronization regions in parameter space.

Let us analyze further the dynamics inside and outside an Arnold tongue. To this end, we use the linear phase $\phi(t)$ defined in Eq. (6) and the phase associated to the driving force, that is, $\phi_D(t)=\omega_D t$. In Fig. 7 we show the difference between these two phases, divided by 2π, for five different ratios ω_D/ω_0, and fixing $F_D=2$ and the tilt $F=1$. These values correspond to five points in the parameter space in Fig. 3, as shown in the inset. Three points are inside the Arnold tongue 1:1, one at the center and the other two at the left and right borders; the other two points are immediately outside the tongue 1:1, one just outside the left border and the other just outside the right border. The phase difference is plotted as a function of time (scaled with the driving period) in Fig. 7. We notice that for the three points inside the tongue this difference is exactly zero, as expected. This means simply that we are in a synchronized region and therefore the two phases are identical, indicating phase synchronization. For the two points just outside the tongue 1:1, we observed the phenomenon of phase slip characterized by a series of steps in the phase difference. Just outside the left border we have a slip to positive values of the difference, and a slip to negative values for the point just outside the right border of the tongue. This behavior is also expected since we are no longer in the region of synchronization and thus the phase difference tends to grow in time, indicating that phase synchronization is lost.

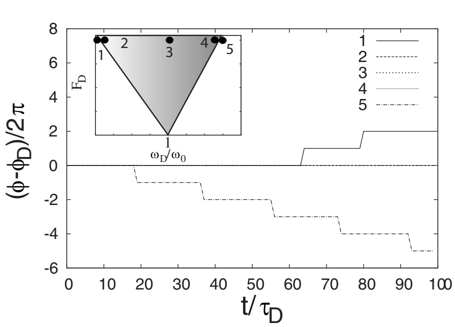

FIG. 7. Phase difference between the linear phase and the phase of the driving force, divided by 2π, as a function of time (scaled with the driving period τ_D). The five curves correspond to five points around the Arnold tongue 1:1 in the parameter space, as indicated in the inset. For the three points inside the tongue the phase difference is exactly zero, corresponding to phase synchronization. For the two points just outside the left and right borders, we observed the phenomenon of phase slip. We fixed $F_D=2$ and the tilt $F=1$, and we used five different ratios ω_D/ω_0, as indicated in the inset.

There is a qualitative difference between the dynamics of an overdamped and an underdamped particle in a tilted rocking ratchet potential. In both cases, the dynamics can become synchronized and we have Arnold tongues in the parameter space. However, in the overdamped case the dynamics is regular and we do not have chaotic motion, since there is no inertial term. On the other hand, in the underdamped case, we have an inertial term in the equation of motion that makes the dynamical system three-dimensional and the possibility of chaotic motion exists. The origin of this chaotic dynamics is related to the overlapping of resonances or Arnold tongues when the driving frequency is above a critical value.[51,73] In order to illustrate the role played by the chaotic dynamics in the current, we calculate a bifurcation diagram for the velocity, as a function of the amplitude of the driving force F_D, in a regime where chaos is present. In Fig. 8 we depict this bifurcation diagram that clearly shows regions of periodic and chaotic motion. Additionally, in the same figure we calculate the scaled average velocity (current) $2\pi\langle v\rangle/\omega_D$ for a frequency ratio $\omega_D/\omega_0=0.7$. With this ratio we are choosing the Arnold tongue 7:10, whose tip is at 0.7 for $F_D=0$. We choose this ratio as a typical example of a tongue that has a small width in the parameter space. By fixing ω_D/ω_0 and varying F_D, we are moving in a vertical line in the parameter space shown in Fig. 3. For small values of F_D we are moving inside the tongue 7:10, but as we increase the value of this control parameter we enter a region where there are overlapping resonances that leads to chaos. In Fig. 8 we notice that the current is approximately constant, with a value $2\pi\langle v\rangle/\omega_D\simeq 1.4$, which correspond to the inverse 10/7 of the rational that characterize the tongue 7:10. This current is nearly constant for a range $0<F_D<1.9$, but for larger values of $F_D\simeq 2.5$ the current starts to fluctuate chaotically until it enters again in a regime of synchronization, but this time with a current equal to 1, for $F_D\simeq 2.75$, that correspond to an Arnold tongue 1:1. This is an indication that the tongue 7:10 overlaps with the tongue 1:1. Thus, this example illustrates

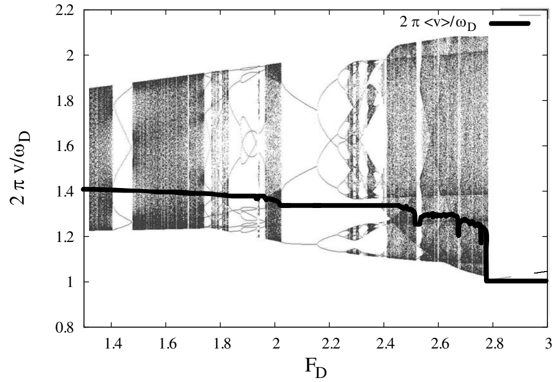

FIG. 8. Bifurcation diagram for the velocity of a particle in the tilted ratchet potential as a function of the control parameter F_D showing chaotic regions and windows of periodicity. Additionally, we indicate as a bold line, the scaled average velocity (current) $2\pi\langle v\rangle/\omega_D$ for a frequency ratio $\omega_D/\omega_0=0.7$. With this ratio we are choosing the Arnold tongue 7:10, whose tip is at 0.7 for $F_D=0$. The current is nearly constant for a range $0<F_D<1.9$, but for larger values of $F_D\simeq 2.5$ the current fluctuates chaotically until it enters again in a regime of synchronization, but this time with a current equal to 1, for $F_D\simeq 2.75$, that correspond to an Arnold tongue 1:1.

the effect of the overlap of resonances and chaos in the current of an inertial ratchet. For the overdamped case discussed in Ref. 71, there are resonances and synchronization, but no chaos, and thus the current is more stable. This is a clear distinction between both cases.

IV. CONCLUDING REMARKS

We have studied the deterministic dynamics of an inertial particle moving in a tilted ratchet potential rocked by a periodic driving force. The value of the tilt is above a critical value such that the particle slides down this washboard potential and in this way can be considered as a rotator or self-sustained oscillator with a characteristic frequency. Furthermore, the frequency of this nonlinear oscillator can be synchronized with the external periodic drive. To analyze this possibility, we define an instantaneous linear phase using a set of discrete time events extracted from the continuous dynamics, and obtained the associated frequency. We showed that this chaotic rotator can exhibit the phenomenon of phase synchronization characterized by Arnold tongues in a two-dimensional parameter space given by the amplitude and the frequency of the driving force. These tongues or resonances can overlap, leading to a chaotic dynamics for this inertial rocking ratchet. We analyzed the consequences of this dynamics for the transport properties by calculating the average velocity (current) as a function of the parameters of the system. We found that this current develops a complex structure due to the chaotic dynamics and also acquires a series of steps given by the widths of the Arnold tongues. Each tongue is labeled by a rational number whose inverse gives the value of the scaled average velocity of a particle in this ratchet. Finally, we show that the local maxima in the current correspond to the borders of these Arnold tongues and, in this way, we established a connection between optimal transport in ratchets and the phenomenon of phase synchronization.

ACKNOWLEDGMENTS

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