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Traversal-time distribution for a classical time-modulated barrier

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Abstract

The classical problem of a time-modulated barrier, inspired by the Büttiker and Landauer model to study the tunneling times, is analyzed. We show that the traversal-time distribution of an ensemble of non-interacting particles that arrives at the oscillating barrier, obeys a distribution with a power-law tail. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The problem of obtaining the time involved in the tunneling process in quantum mechanics is still a controversial issue, despite considerable efforts in recent years [1]. In particular, in order to address this issue, some authors have analyzed the tunneling through time-modulated potential barriers [2–14]. One of the pioneer works in this area is the model introduced by Büttiker and Landauer in 1982 [2] in which they consider the transmission through a time-modulated rectangular barrier, and introduced a characteristic time for the process. However, in the above-mentioned papers, there is practically no mention of the corresponding classical problem; although the classical limit is straightforward when the poten-

tial barrier does not depend on time, it is far from trivial when the potential is time modulated.

In this paper I study the classical problem of a rectangular time-modulated potential barrier, in order to analyze in detail the traversal time distribution for an ensemble of classical particles. This classical model was inspired, in part, by the Büttiker-Landauer model mentioned above.

I will study first the case of a potential barrier located inside a rigid box [15]. In this case, the classical orbits can be periodic, quasiperiodic or even chaotic, depending on the parameters and the initial conditions of the motion. In order to study the dynamics, I derive first an area-preserving map that allow us to find the orbits for all times. Then, I study the scattering problem of an ensemble of particles that interact with an oscillating rectangular potential barrier. In this case, I will show that the traversal time strongly depends on the arrival time of the incident particles.

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There is a basic difference between these two problems: (1) In the first case, what we have is the bounded problem of an oscillating barrier inside a rigid box of finite size. This means that an incident particle interact with the barrier not once but an arbitrary number of times, since the particle can cross the barrier region and then, after bouncing elastically in the box, returns to the oscillating barrier. Then, the dynamics can become chaotic, since we have the main ingredients: on one hand, sensitive dependence on initial condition or arrival times due to the oscillating barrier, and on the other hand, bounded motion due to presence of the finite box. (2) In the second case, we have a scattering problem in which an incident particle interacts with the barrier only once. Of course, if this is the case, the problem is straightforward and there is only a single traversal time. But, if we consider an ensemble of N noninteracting particles with slightly different initial conditions, say different initial velocities, then we can expect, in general, N different traversal times that can exhibit a complex distribution of traversal times.

An approach to the problem of tunnelling times, that is closely related to the classical trajectories discussed here, is the Bohm trajectory point of view [16]. This approach has been used by Leavens and Aers [17] to give an unambiguous prescription for calculating traversal times that are conceptually meaningful within that interpretation. In particular, Leavens and Aers [6,7] have treated in detail the case of a time-modulated rectangular barrier, using Bohm's trajectory interpretation of quantum mechanics [16]. They calculate, among other things, transmission time distributions, the transmission probability as a function of frequency and Bohm trajectories.

2. The model and the map

Let us study the classical dynamics of a particle in a one-dimensional box, inside of which there is an oscillating rectangular potential barrier [15]. This problem consists of a particle moving in one dimension under the action of a time-dependent potential $V(x,t)$. Since the Hamiltonian of this system is time dependent, the total energy of the particle is not

conserved. The Hamiltonian is given by $H(x,p,t) = p^2/2m + V(x,t)$, where

$$V(x,t) = V_0(x) + V_1(x)f(t). \quad (1)$$

The potential $V_0(x)$ goes to infinity when $x < 0$ or $x > l + b + L$, is equal to the constant value V_0 when $l \leq x \leq l + b$, and otherwise is equal to zero. Thus, what we have is an infinite potential well with a rectangular potential barrier of width b inside, as shown in Fig. 1a. This potential separates the box in three regions: region I, $0 \leq x < l$ of width l ; region II, where the rectangular barrier is located, $l \leq x \leq l + b$ of width b ; and region III, $l + b < x \leq l + b + L$ of width L .

Clearly, the motion of a particle under the influence of the potential $V_0(x)$ is regular, that is, we

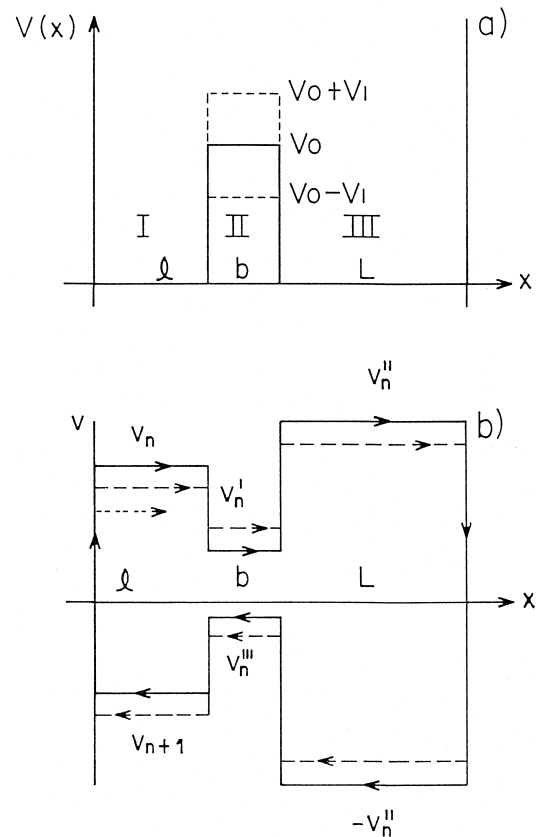


Fig. 1. a) Potential well with a rectangular time-modulated potential barrier of width b . The height of the barrier oscillates harmonically between $V_0 + V_1$ and $V_0 - V_1$. b) Typical orbit in phase space, showing a general change in the velocity for one iteration of the map (solid line) and a second iteration (dashed line).

have periodic orbits and the energy is conserved. However, if we add a time-dependent potential, we can obtain periodic, quasiperiodic and chaotic orbits, as we will show below. The potential $V_1(x)$ in Eq. (1) is different from zero only inside the interval $l \leq x \leq l + b$, where it takes the constant value V_1 . The function $f(t)$ in Eq. (1) is assumed periodic with period τ , that is, $f(t + \tau) = f(t)$. In this way, as shown in Fig. 1a, what we have is an oscillating potential barrier, with an amplitude which oscillates between $V_0 - V_1$ and $V_0 + V_1$, with frequency $\omega/2\pi$ and period $\tau = 2\pi/\omega$. We will take $V_0 > V_1$.

Let us now derive a map that describes the dynamics of a particle under this potential. The motion is as follows: at the fixed walls at $x = 0$ and $x = l + b + L$, the particle bounces elastically, changing the sign of the velocity but with the same absolute value. The other two points where there is a change in the velocity is at the borders of the potential barrier at $x = l$ and $x = l + b$. The rest of the time the velocity is constant. Thus, the particle can gain or lose kinetic energy at $x = l$ and $x = l + b$. The phase space for a typical orbit is depicted in Fig. 1b.

We can analyze the dynamics using a discrete map from the time t_n when the particle hits the wall at $x = 0$, until the next time t_{n+1} when it hits this wall again. Let us denote by v_n the velocity of the particle immediately after the n -th kick with the fixed wall at $x = 0$, and by E_n the corresponding total energy. Clearly, $E_n = mv_n^2/2$. After traveling the distance l , it arrives at the left side of the barrier after a time of flight l/v_n , where a change in the velocity occurs. To determine this change let us consider the following: In region I, the total energy of the particle is given by $E_n = mv_n^2/2$ which is just the kinetic energy, because in this region the potential energy is zero; when the particle enters region II, the kinetic energy E_n is changed to $E_n - V_0 - V_1 f(t_n + l/v_n)$, that is, the total energy minus the value of the potential energy at the time of arrival $t_n + l/v_n$. If we denote the new velocity by v'_n (see Fig. 1b), then $E_n = mv_n^2/2$ and we obtain in this way the change in energy as

$$E'_n = E_n - V_0 - V_1 f\left(t_n + \frac{l}{v_n}\right). \quad (2)$$

Clearly, if the total energy is less than the potential energy at time $t_n + l/v_n$, then the particle cannot

penetrate region II and simply reflects elastically and there is only a change in the sign of the velocity; thus the particle gets trapped in region I and returns to the wall at $x = 0$. After a time lapse of $2l/v_n$ it will hit again the oscillating barrier and try again to cross it. If this time the total energy is greater than the potential energy, then the particle can cross the barrier region; otherwise, it bounces once more inside region I, and so on.

Now, once the particle overcomes the barrier, it crosses the region II without changing its velocity v'_n , even though the barrier is oscillating in time. When the particle arrives at the right side of the barrier at $x = l + b$, then another change in the velocity takes place, but this time the velocity increases in such a way that the kinetic energy E''_n becomes

$$E''_n = E'_n + V_0 + V_1 f\left(t_n + \frac{l}{v_n} + \frac{b}{v'_n}\right), \quad (3)$$

where E''_n is the energy in region III. Clearly, the time that it takes to arrive at the wall located at $x = l + b + L$ is $l/v_n + b/v'_n + L/v''_n$, where v''_n is the velocity in region III (see Fig. 1b). After a time $t_n + l/v_n + b/v'_n + 2L/v''_n$, the particle returns to the right side of the barrier after traveling twice the distance L in region III, and enters once again region II. However, in general, the potential barrier has a different height, given by $V_0 + V_1 f(t_n + l/v_n + b/v'_n + 2L/v''_n)$. Therefore, the new kinetic energy E'''_n inside region II is now given by

$$E'''_n = E''_n - V_0 - V_1 f\left(t_n + \frac{l}{v_n} + \frac{b}{v'_n} + \frac{2L}{v''_n}\right). \quad (4)$$

Here, once more, there is the possibility that the total energy in region III is less than the potential energy at time $t_n + l/v_n + b/v'_n + 2L/v''_n$. In this case, the particle gets trapped in region III until it can escape by crossing the barrier region.

Finally, after a time $b/|v'''_n|$, where v'''_n is the velocity in region II (see Fig. 1b), the particle arrives at the left side of the barrier at $x = l$, where the velocity varies once more depending on the height of the barrier at time $t_n + l/v_n + b/v'_n + 2L/v''_n + b/|v'''_n|$. We will denote the velocity in region I, after this time, by v_{n+1} , because this is precisely the velocity after the next hit with the wall at $x = 0$. The

last part of this journey is covered in a time span of $l/|v_{n+1}|$; after this, the particle hits the wall at the origin at time t_{n+1} and start again its trip to the oscillating barrier, and the whole process starts again.

Therefore we arrive at the following map in terms of energy and time:

$$E_{n+1} = E_n''' + V_0 + V_1 f(t_n + T_n) \quad (5)$$

and

$$t_{n+1} = t_n + T_n + \sqrt{\frac{m}{2}} \frac{l}{\sqrt{E_{n+1}}}, \quad (6)$$

where T_n is given by

$$T_n = \sqrt{\frac{m}{2}} \left(\frac{l}{\sqrt{E_n}} + \frac{b}{\sqrt{E_n'}} + \frac{2L}{\sqrt{E_n''}} + \frac{b}{\sqrt{E_n'''}} \right) \quad (7)$$

and E_n , E_n' and E_n'' are given by Eqs. (2)–(4), respectively.

Furthermore, it can be shown that, for this map, the Jacobian is exactly one, that is,

$$J = \frac{\partial(E_{n+1}, t_{n+1})}{\partial(E_n, t_n)} = 1. \quad (8)$$

This result indicates that this map is an area-preserving one [18].

Let us scale the time using the period τ of the function $f(t)$. We define the dimensionless quantities: $\phi_n = (2\pi/\tau)t_n$ and $\Phi_n = (2\pi/\tau)T_n$. In order to scale the energies we introduce the dimensionless variables: $e_n = E_n/V_0$, $e_n' = E_n'/V_0$, $e_n'' = E_n''/V_0$ and $e_n''' = E_n'''/V_0$. With all this definitions we arrive at the following dimensionless map:

$$e_{n+1} = e_n''' + 1 + rf(\phi_n + \Phi_n), \quad (9)$$

and

$$\phi_{n+1} = \phi_n + \Phi_n + \frac{2\pi M}{\sqrt{e_{n+1}}}, \quad (\text{mod } 2\pi) \quad (10)$$

where $M = l/(w\tau)$, $r = V_1/V_0$ and $w = (2V_0/m)^{1/2}$. Here, Φ_n is given by

$$\Phi_n = 2\pi M \left(\frac{1}{\sqrt{e_n}} + \frac{b}{l} \frac{1}{\sqrt{e_n'}} + \frac{2L}{l} \frac{1}{\sqrt{e_n''}} + \frac{b}{l} \frac{1}{\sqrt{e_n'''}} \right). \quad (11)$$

This map, although more complicated, resembles the structure of the Fermi Map [18].

3. Numerical results

Let us now analyze numerically the map obtained above. First of all, we notice that we have four dimensionless parameters: the width of the barrier b/l scaled with the length of region I; the length L/l of region III scaled with l ; the ratio of the amplitude of oscillation of the barrier scaled with its height $r = V_1/V_0$; and $M = l/(w\tau)$. The parameter M is the ratio of the time of flight l/w in region I of Fig. 1a, with velocity w , and the period τ of oscillation of the barrier. That is, M measures the number of oscillations of the barrier since the particle leaves the wall at $x = 0$ until it arrives at the left side of the barrier. On the other hand, we will take the periodic function as: $f(\phi_n) = \sin(\phi_n)$.

If we fix the barrier position within the one-dimensional box, and choose a width, then we are fixing the parameters b/l and L/l ; the remaining two parameters M and r will control the type of motion. In what follows, we take the symmetric case, $b/l = 1$ and $L/l = 1$, which corresponds to the oscillating barrier centered inside the box, and an oscillating amplitude of $r = 0.5$.

In Fig. 2 we show the energy-phase space (e_n, ϕ_n) for $M = 4.7$, using the map given by Eqs. (9)–(11). We plot several orbits that correspond to different initial conditions. We can clearly see that, for this system, we have a phase space with a mixed structure, in which we have periodic, quasiperiodic and chaotic orbits. Some of the fixed points of the map can be seen surrounded by elliptic orbits. We notice a fine structure of smaller islands in the chaotic region, as is usually the case for other maps [18].

The quantity that we want to analyze in detail is the traversal time in the barrier region, that is, the time it takes the particle to cross the region where the barrier is oscillating. We can obtain this quantity simply as b/v_n' or $b/|v_n'''|$ (see Fig. 1b). The structure of this traversal or dwell time depends strongly on the type of orbit. Clearly, if we have a periodic orbit, then this time will take only two possible values, since v_n' and v_n''' does not change with n . On the other hand, if the orbit is quasiperiodic, the

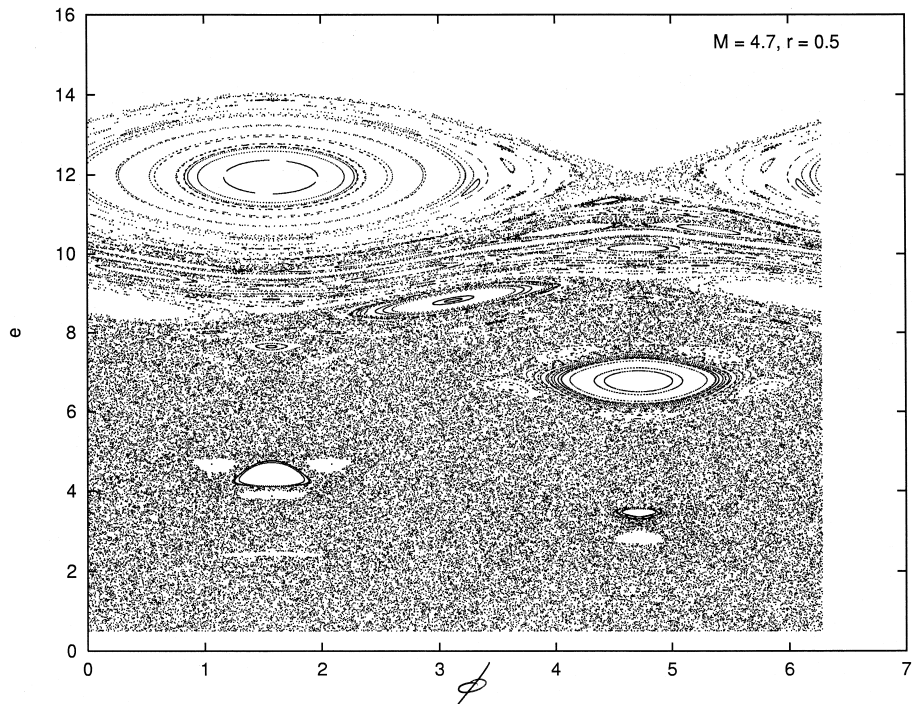


Fig. 2. Phase space (e_n, ϕ_n) , for $M = 4.7$ and $r = 0.5$, showing periodic, quasiperiodic and chaotic orbits for different initial conditions.

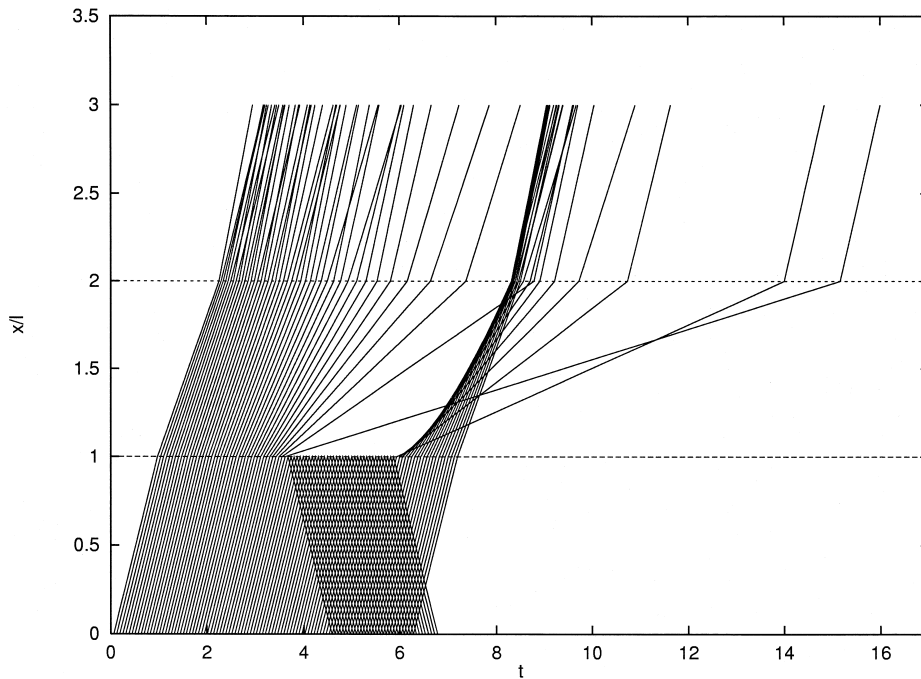


Fig. 3. Space-time diagram of trajectories for an ensemble of incident particles with the same velocity and different phases. In this case $M = 77.7$ and $r = 0.5$. The horizontal dashed lines indicate the barrier region.

velocity can vary in a full range of values. In this case, the traversal time can vary only in a limited range. However, when we have a chaotic orbit, the variation can display a very rich structure [15].

For the bounded problem, where the oscillating barrier is confined within a box, we can obtain a chaotic dynamics as shown in Fig. 2. However, if we remove the walls and leave only the oscillating barrier, we end up with an open system of the scattering type. In this case, we cannot have chaotic dynamics, since the particle interacts with the barrier only once. However, we can study not a single particle, but an ensemble of noninteracting particles, each of them with different initial conditions.

In Fig. 3 we show a space-time diagram of trajectories for an ensemble of incident particles. In this case, and for the rest of the figures, we take $r = 0.5$ and $M = 77.7$. I use dimensionless distance x/l and dimensionless time t , which is the time scaled with l/w . Since $l = b$, then l/w is the time it takes to cross the barrier region with a velocity $w = (2V_0/m)^{1/2}$. The barrier is located between $x/l = 1$ and $x/l = 2$, and is indicated by horizontal dashed lines in Fig. 3. We take an ensemble of initial

conditions in which the initial velocity is constant and the initial phase is uniformly distributed. We see from Fig. 3 that only a subset of particles in the ensemble can cross the barrier region and that the traversal time is different for each particle. This is due to the fact that each particle is influenced differently by the time-modulated barrier, depending on the arrival time. That is, different arrival times mean different barrier amplitudes.

The traversal time is defined as the time it takes to cross the region where the barrier is oscillating, and is given by b/v'_n . Since we scale this traversal time with the time b/w , the dimensionless form is given by $1/\sqrt{e'_n}$. For the particles in the ensemble, this time is shown in Fig. 4. We notice that in many cases the dimensionless time $t \sim 1$; however, there are some others cases for which $t \gg 1$. These large peaks occur when the arrival time is such that the total energy is just above the barrier height, and thus the velocity inside the barrier region is very small and consequently the traversal time is very large. We can see a strong variation in the traversal time, that leads to a broad distribution of times. On the other hand, since the minimum velocity in the barrier

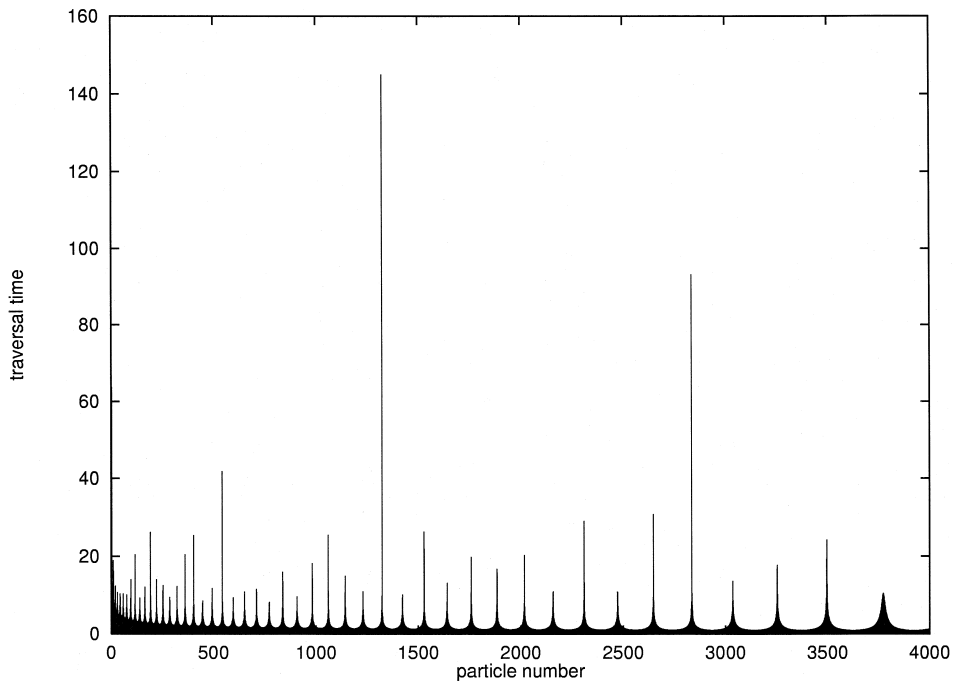


Fig. 4. Traversal time for an ensemble of incident particles. In this case $M = 77.7$ and $r = 0.5$.

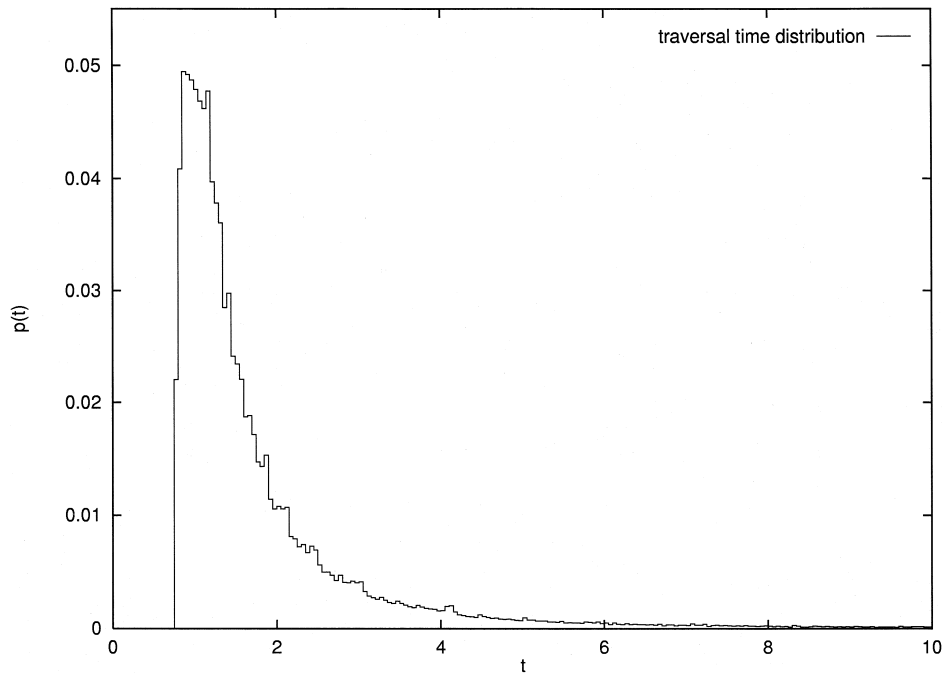


Fig. 5. Traversal time distribution for the case $M = 77.7$ and $r = 0.5$.

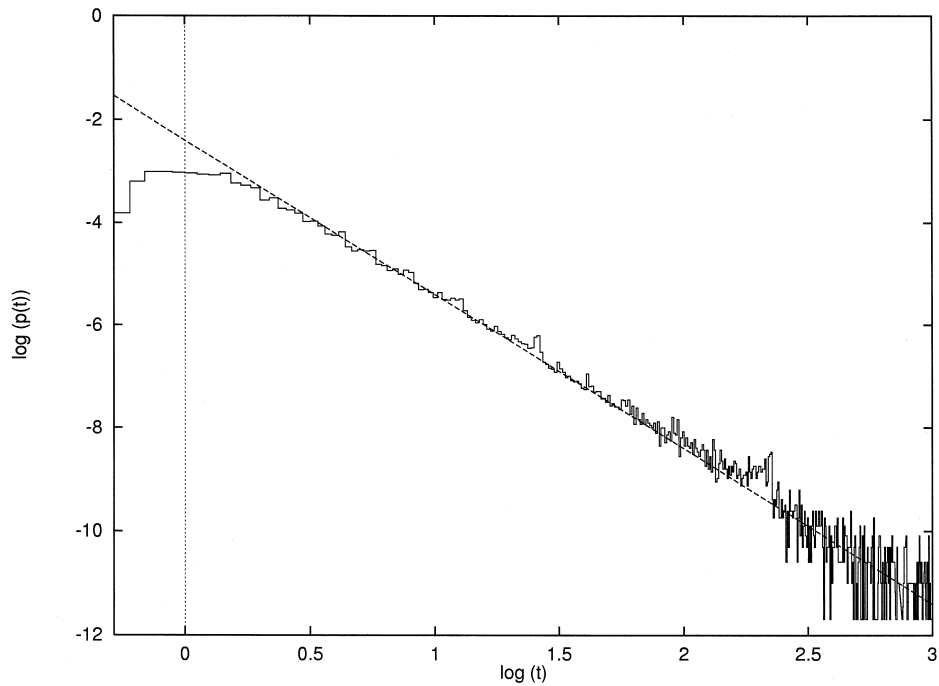


Fig. 6. Log-log plot of the traversal time distribution of Fig. 5, clearly showing a power law. The slope of the dashed line is -3 .

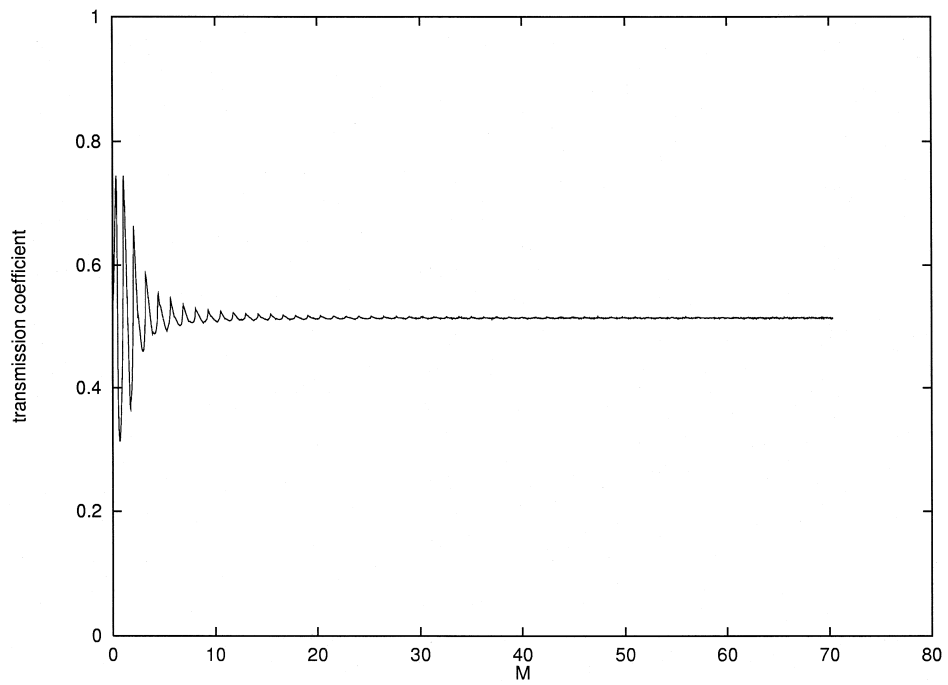


Fig. 7. Transmission coefficient as a function of M , for $r = 0.5$.

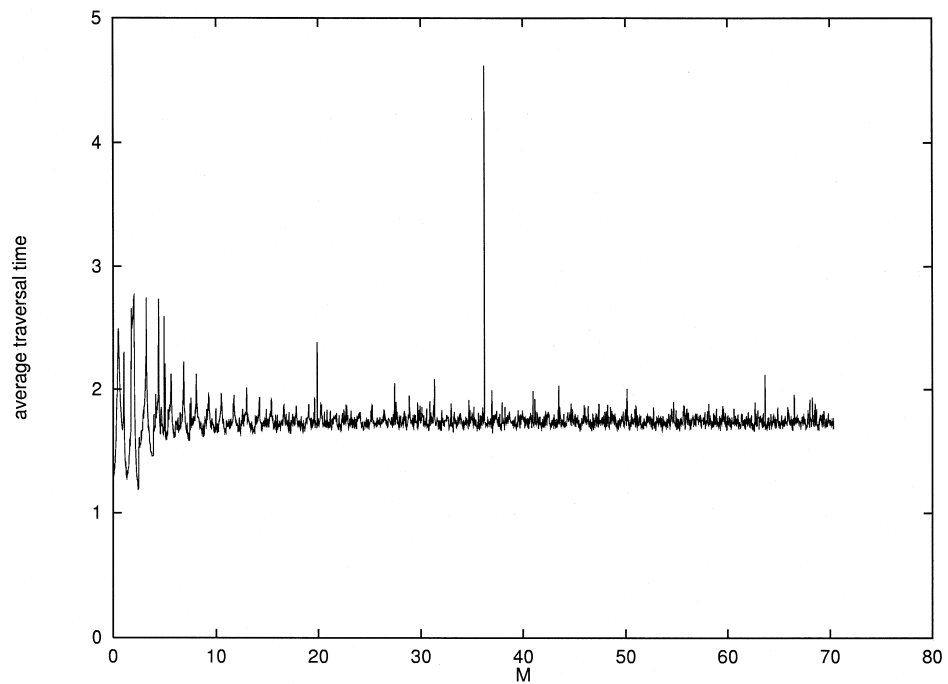


Fig. 8. Average traversal time as a function of M , for $r = 0.5$.

region is zero, then there is no upper bound for the dwell time, and it can acquire very large values, as seen in Fig. 4.

The traversal time distribution is depicted in Fig. 5. This normalized distribution has a long-time tail which is a power law. In Fig. 6 we show the same distribution in a log-log plot that clearly shows that this is indeed a distribution with a power-law tail of the form $p(t) \sim t^{-\alpha}$, with $\alpha \approx 3$. The straight (dashed) line in this figure has a slope of -3 .

Another quantity of interest is the transmission coefficient, defined as the number of particles that cross the barrier region, divided by the total number of particles in the ensemble. In Fig. 7 we show this transmission coefficient as a function of M . Remember that $M = l/(w\tau)$ and is, therefore, proportional to the frequency of oscillation of the barrier. We can see in this figure that the transmission coefficient vary strongly with M , in particular for low frequencies ($M \sim 1$). On the other hand, for higher frequencies ($M \gg 1$), the transmission coefficient tend towards a constant value. This last result indicates that for $M \gg 1$, the oscillating potential barrier acts as an effective potential barrier of average height V_0 .

Finally, in Fig. 8 we show the average traversal time as a function of M . Again we can see strong fluctuations of this quantity. Since the distribution of traversal times is a power law with an exponent $\alpha \approx 3$, we can expect these large fluctuations; although the first moment of the distribution is finite in this case, the second or higher moments can diverge, leading to these large fluctuations, as is usually the case for Lévy distributions [19].

4. Concluding remarks

In this paper, the dynamics of the classical problem of an oscillating rectangular potential barrier is analyzed. When the oscillating barrier is located within a one-dimensional box, we have a bounded problem and the corresponding classical dynamics can have a mixed phase space structure comprising periodic, quasiperiodic and chaotic orbits. For the scattering problem of a single oscillating barrier, a distribution of traversal times with a power-law tail is obtained. This Lévy-type distribution of times leads to large fluctuations of the average traversal time as a function of the frequency of oscillation of

the barrier; therefore, it is difficult to obtain a characteristic time to the process of crossing the classical oscillating barrier. These large fluctuations arise due to the sensitive dependence on initial conditions, typical of the dynamics of chaotic systems. In particular, for our problem, the quantity that controls the traversal time is the time of arrival at the barrier. Thus, we obtain a sensitive dependence on the time of arrival for the classical case. The possible role for the tunneling time problem, if any, of the sensitive dependence on the time of arrival and the difficulty to obtain a characteristic traversal time in the classical domain, remains to be seen.

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